On the Minimum Spectral Radius of Matrices of Zeros and Ones*

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ABSTRACT

We consider the minimum spectral radius for an $n \times n$ matrix of 0's and 1's having a specified number τ of 0's. We determine this minimum spectral radius when $\tau \leqslant \lfloor n/2 \rfloor \lceil n/2 \rceil$, and bound it between two consecutive integers for all other values of τ.

1. INTRODUCTION

In a previous paper [3] we formulated the general problem of determining the minimum and maximum spectral radius of a given class of matrices of 0's and 1's. The maximum spectral radius problem was investigated by Brualdi and Hoffman [2] and Friedland [4] for the class of matrices of 0's and 1's with a prescribed number of 1's. In addition we determined in [3] the minimum and maximum spectral radius for the class of $n \times n$ complementary acyclic matrices. In this paper we consider the minimum spectral radius problem for the class of $n \times n$ matrices of 0's and 1's with a prescribed number τ of 0's. When

$$\tau \geqslant \binom{n+1}{2},$$

the minimum spectral radius is clearly 0. For

$$\binom{n}{2} \leqslant \tau < \binom{n+1}{2},$$

*Research partially supported by National Science Foundation Grant No. DMS-8320189.

52 Vanderbilt Ave., New York, NY 10017 0024-3795/87/$3.50

the minimum spectral radius is easily seen to be 1. We determine the minimum spectral radius when $\tau \leqslant \lfloor n/2 \rfloor \lceil n/2 \rceil$ and characterize those matrices whose spectral radius attains the minimum value. We also determine the minimum spectral radius for a certain sequence of values of τ and then use this result to bound the minimum spectral radius between two consecutive integers.

We now discuss briefly those parts of the Perron-Frobenius theory [1,5] of nonnegative matrices that we make use of. Let $B=[b_{ij}]$ be an $n \times n$ nonnegative matrix. The *spectral radius* of B, that is, the maximum absolute value of an eigenvalue of B, is denoted by $\rho(B)$. Recall that B is *reducible* if there exists a permutation matrix P such that

$$P'BP = \begin{bmatrix} B_1 & 0 \\ B_{21} & B_2 \end{bmatrix},$$

where B_1 and B_2 are square, nonvacuous matrices; B is *irreducible* when it is not reducible. If $C=[c_{ij}]$ is another $n \times n$ nonnegative matrix, we write $B \leqslant C$ when $b_{ij} \leqslant c_{ij}$ for all i and j. From the Perron-Frobenius theory we have the following:

(1.1) $\rho(B)$ is an eigenvalue of B with an associated nonnegative eigenvector u (when B is irreducible, u is positive).

(1.2) If B has row sums $r_1, \dots, r_n$, then

$$\min\{r_1,\dots,r_n\} \leqslant \rho(B) \leqslant \max\{r_1,\dots,r_n\}.$$

When B is irreducible and not all row sums are equal, both of the inequalities are strict.

(1.3) Let z be a positive vector. If $Bz \geqslant rz$ [respectively, $Bz \leqslant rz$], then $\rho(B) \geqslant r$ [respectively, $\rho(B) \leqslant r$] with equality for irreducible B if and only if $Bz = rz$.

(1.4) If $B \leqslant C$, then $\rho(B) \leqslant \rho(C)$ with strict inequality when C is irreducible and $B \neq C$.

(1.5) If B' is a proper principal submatrix of B, then $\rho(B') \leqslant \rho(B)$ with strict inequality when B is irreducible.

Let n be a positive integer, and let τ be an integer with $0 \leqslant \tau \leqslant n^2$. Denote the class of all $n \times n$ matrices of 0's and 1's with exactly τ 0's by $L(n, \tau)$. Let

$$\tilde{\rho}(n, \tau) = \min\{\rho(A) : A \in L(n, \tau)\}.$$

Let $\tilde{L}(n, \tau)$ denote the subset of $L(n, \tau)$ consisting of those matrices having the property that in each row the 1's are to the left of the 0's and in each column the 0's are above the 1's. For example, the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

is in $\tilde{L}(5,8)$. Let $A \in \tilde{L}(n, \tau)$. Then it follows easily by induction that the determinant of A equals 0 or 1. Since a principal submatrix of A of order m is in $\tilde{L}(m, \tau')$ for some τ', each principal submatrix of A also has determinant equal to 0 or 1.

We now discuss briefly a theorem of B. Schwarz [6] as it pertains to $L(n, \tau)$ and $\tilde{L}(n, \tau)$ along with the basic idea in its proof.

(1.6) Let $A \in L(n, \tau)$. Then for some permutation matrix Q, there exists a sequence of matrices $A_0 = Q'AQ$, $A_1, \dots, A_s = B$ such that

(i) $B \in \tilde{L}(n, \tau)$,

(ii) A_{i+1} is obtained from A_i by switching a 0 and a 1 in A_i where the 0 immediately precedes the 1 in some row or immediately follows the 1 in some column $(i = 0, 1, \dots, s-1)$,

(iii) $\rho(A) \geqslant \rho(A_i) \geqslant \rho(B)$ $(i = 1, \dots, s-1)$.

The sequence of matrices in (1.6) can be chosen so that we move the 0's to the right in each row and then to the top in each column in the manner prescribed by (ii). Schwarz's argument in [6] was given for positive matrices, but (1.6) follows by a continuity argument after replacing the 0's of A by ε where $0 < \varepsilon < 1$.

A consequence of (1.6) is the following:

$$\tilde{\rho}(n,\tau) = \min\{\rho(A) : A \in \tilde{L}(n,\tau)\}. \tag{1.7}$$

In words, the minimum spectral radius of matrices in $L(n,\tau)$ is achieved by a matrix in $\tilde{L}(n,\tau)$.

We conclude this introduction by mentioning some notation. Let A be an $n \times n$ matrix, let $i_1,\dots,i_k$ be integers where $1 \leqslant i_1 < \cdots < i_k \leqslant n$, and let $j_1,\dots,j_l$ be integers where $1 \leqslant j_1 < \cdots < j_l \leqslant n$. Then $A[i_1,\dots,i_k|j_1,\dots,j_l]$ denotes the $k \times l$ submatrix of A formed by rows $i_1,\dots,i_k$ and columns $j_1,\dots,j_l$. The matrix $A(i_1,\dots,i_k|j_1,\dots,j_l)$ is the $(n-k)\times(n-l)$ submatrix obtained by deleting rows $i_1,\dots,i_k$ and columns $j_1,\dots,j_l$. For integers k and l, $J_{k,l}$ denotes the $k \times l$ matrix of all 1's. When k or l is 0, $J_{k,l}$ is an empty matrix. When $k = l$, we write J_k instead of $J_{k,k}$.

2. EVALUATION OF THE MINIMUM SPECTRAL RADIUS FOR AT MOST $\lfloor n/2 \rfloor \lceil n/2 \rceil$ 0's

We assume in this section that $n \geqslant 2$. Our purpose is to prove the following result.

THEOREM 2.1. *Let τ be an integer with $0 \leqslant \tau \leqslant \lfloor n/2 \rfloor \lceil n/2 \rceil$. Then*

$$\tilde{\rho}(n,\tau) = \tfrac{1}{2}\left(n + \sqrt{n^2 - 4\tau}\right).$$

Moreover, for $A \in L(n,\tau)$, $\rho(A) = \tilde{\rho}(n,\tau)$ if and only if there is a permutation matrix P and nonnegative integers k and l with $k + l = n$ such that P^tAP has the form

$$\begin{bmatrix} J_k & X \\ J_{l,k} & J_l \end{bmatrix}. \tag{2.1}$$

COROLLARY 2.2. *Let r be an integer with $0 \leqslant r \leqslant n$, and let $\tau = r(n-r)$. For $A \in L(n,\tau)$,*

$$\rho(A) \geqslant \max\{r, n-r\} \tag{2.2}$$

with equality if and only if there is a permutation matrix P and nonnegative integers k and l with $k + l = n$ such that (2.1) *holds.*

For $\tau = r(n-r)$, equality holds in (2.2) when A has the form (2.1) with $k = r$ or $k = n - r$, in which case the matrix X in the upper right corner is a zero matrix. But in general there are other values of k and l for which the matrix in (2.1) gives equality. For example, let $n = 6$ and $r = 2$, so that $\tau = 8$. Then the matrix

$$B = \begin{bmatrix} J_3 & X \\ J_3 & J_3 \end{bmatrix},$$

where X has one 1 and eight 0's, satisfies $\rho(B) = \tilde{\rho}(6,8) = 4$.

We denote the characteristic polynomial of a matrix A by

$$\chi_A(\lambda) = \lambda^n - c_1\lambda^{n-1} + c_2\lambda^{n-2} - c_3\lambda^{n-3} + \cdots + (-1)^n c_n.$$

Suppose $A \in \tilde{L}(n,\tau)$ for some n and τ. Since the determinant of a principal submatrix of A equals 0 or 1, c_m equals the number of nonsingular principal submatrices of A of order m. In particular, $c_m \geqslant 0$ for $m = 1, \dots, n$.

We now prove a sequence of lemmas which will be used in the proof of Theorem 2.1.

LEMMA 2.3. *Let $A \in L(n,\tau)$ have the form* (2.1). *Then the characteristic polynomial of A is*

$$\chi_A(\lambda) = \lambda^{n-2}(\lambda^2 - n\lambda + \tau),$$

and hence $\rho(A) = \frac{1}{2}(n + \sqrt{n^2 - 4\tau})$.

Proof. Since the trace of A is n, $c_1 = n$. A 2×2 principal submatrix of A has a nonzero determinant if and only if it equals

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Since there are exactly τ principal submatrices of this form, it follows that $c_2 = \tau$. A principal submatrix of A of order at least 3 either has two rows or two columns of all 1's, and it follows that $c_i = 0$ for $i = 3, \dots, n$. ■

For τ an integer with $0 \leqslant \tau \leqslant \lfloor n/2 \rfloor \lceil n/2 \rceil$, we put $\lambda(n,\tau) = (n + \sqrt{n^2 - 4\tau})/2$. We remark that since $\tau \leqslant \lfloor n/2 \rfloor \lceil n/2 \rceil$, there are matrices in $L(n,\tau)$ of the form (2.1) and hence by Lemma 2.3 matrices in $L(n,\tau)$ with spectral radius equal to $\lambda(n,\tau)$.

LEMMA 2.4. *Let $A = [a_{ij}] \in \tilde{L}(n,\tau)$ for some n and τ. Suppose the trace of A equals n and the 0's of A are not confined to any of the submatrices $A[1,\dots,k|k+1,\dots,n]$ for $k = 1,\dots,n-1$. Then $c_3 > 0$.*

Proof. Since $A \in \tilde{L}(n,\tau)$, there is an integer k such that $a_{11} = \cdots = a_{1k} = 1$ and $a_{1,k+1} = \cdots = a_{1n} = 0$. By hypothesis, $A[1,\dots,k|k+1,\dots,n]$ does not contain all the 0's of A, so that there exist integers j and l with $k < j < l$ and $a_{jl} = 0$. Since $A \in \tilde{L}(n,\tau)$, we conclude that

$$A[1,j,l|1,j,l] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Since each principal submatrix of A has determinant equal to 0 or 1, it follows that $c_3 > 0$. ■

LEMMA 2.5. *Let $A = [a_{ij}] \in \tilde{L}(n,\tau)$ for some n and τ, and let r_k be the number of 0's in row k of A for $k = 1,\dots,n$. Suppose the trace of A equals n. Then $c_{k+1} \leqslant r_k c_k$ for $k = 1,\dots,n-1$.*

Proof. Let $1 \leqslant k \leqslant n-1$, and consider any nonsingular principal submatrix $B = A[i_1,\dots,i_k|i_1,\dots,i_k]$ of A of order k. Since A has r_{i_k} 0's in row i_k and since $A \in \tilde{L}(n,\tau)$, there are r_{i_k} integers $i_{k+1} > i_k$ such that $A[i_1,\dots,i_k,i_{k+1}|i_1,\dots,i_k,i_{k+1}]$ is a nonsingular principal submatrix of A of order $k+1$. Since every such matrix of order $k+1$ arises this way and since $r_{i_k} \leqslant r_k$, it follows that $c_{k+1} \leqslant r_k c_k$. ■

LEMMA 2.6. *Let $\tau = r(n-r)$ for some integer r where $2 \leqslant r \leqslant n-1$ and $n - r \geqslant r$. Let $A \in \tilde{L}(n,\tau)$, and suppose that A does not have the form (2.1) for any nonnegative integers k and l with $k + l = n$. Then $\rho(A) > n - r$.*

Proof. We prove the lemma by induction on $n \geqslant 2$. A matrix A satisfying the hypotheses must have order $n \geqslant 4$. Let A be a matrix of order $n = 4$

satisfying the hypotheses. Then $r = 2$ and $\tau = 4$, and A is one of

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

The first and last matrices have spectral radius equal to $3 > 2$, while the second and third have spectral radius equal to $(3+\sqrt{5})/2 > 2$. Hence the conclusion holds for $n = 4$. Now assume that $n > 4$.

Case 1. A has exactly $n - r$ 0's in row 1. The principal submatrix $A(1|1)$ has exactly $(r-1)(n-r) = (r-1)[(n-1)-(r-1)]$ 0's. If $A(1|1)$ has the form (2.1) with $k + l = n - 1$, then $\rho(A(1|1)) = n - r$ by Lemma 2.3. Otherwise, $\rho(A(1|1)) > n - r$ by the inductive assumption. It now follows from (1.5) that $\rho(A) \geqslant n - r$ with equality only if $A(1|1)$ has the form (2.1) and A is reducible. Suppose that A is reducible and

$$A(1|1) = \begin{bmatrix} J_k & X \\ J_{n-1-k,k} & J_{n-1-k} \end{bmatrix}$$

for some k with $0 \leqslant k \leqslant n - 1$. It follows from Lemma 2.3 that $\rho(A(1|1) = n - r$. Since A is reducible and $r \geqslant 2$, it follows that $X = 0$ and hence

$$n - r = \rho(A(1|1)) = \max\{k, n-1-k\}. \tag{2.3}$$

Since A does not have the form (2.1), $n - r > n - 1 - k$. Hence by (2.3), $n - r = k$. Since $r \geqslant 2$, $A[1,\ldots,k+1|1,\ldots,k+1]$ is an irreducible matrix having column sums $s_1,\ldots,s_{k+1}$ where $s_i \geqslant k$ $(i = 1,\ldots,k+1)$ with strict inequality for at least the first two column sums. It now follows from (1.2) and (1.5) that

$$\rho(A) \geqslant \rho(A[1,\ldots,k+1|1,\ldots,k+1]) > k = n - r.$$

Hence $\rho(A) > n - r$ always holds in this case.

Case 2. A has exactly $n - r$ 0's in column n. An argument like the above also shows that $\rho(A) > n - r$.

Case 3. A has more than $n-r$ 0's in row 1. The matrix $A(1|1)$ now has less than $(r-1)(n-r)$ 0's. There is a matrix $A' \in \tilde{L}(n-1,(r-1)(n-r))$ such that $A' \leqslant A(1|1)$, $A' \neq A(1|1)$. If A' has the form (2.1) with $k+l=n-1$, then $\rho(A')=n-r$ by Lemma 2.3. It now follows from the inductive assumption, (1.5), and (1.4) that $\rho(A) \geqslant n-r$. Equality can hold only when A' has the form (2.1) and both A and $A(1|1)$ are reducible. But it is easy to see that these conditions cannot be met when $A' \neq A(1|1)$. Therefore $\rho(A) > n-r$ in this case also.

Case 4. A has more than $n-r$ 0's in column n. An argument similar to the preceding one shows that $\rho(A) > n-r$.

Case 5. A has less than $n-r$ 0's in row 1 and in column n. We first show that the trace of A equals n. Suppose to the contrary $a_{tt}=0$. Since $A \in \tilde{L}(n,\tau)$, $A[1,\dots,t|t,\dots,n]=0$. Since A has at most $n-r-1$ 0's in row 1, $t \geqslant r+1$. We write $t=r+k$ where $k \geqslant 1$. The number of 0's in $A[1,\dots,t|t,\dots,n]$ is

$$(r+k)[n-(r+k-1)] = r(n-r)+k(n-r)-(r+k)(k-1).$$

Since column n has at most $n-r-1$ 0's, $r+k \leqslant n-r-1$ and hence

$$\begin{aligned} k(n-r)-(r+k)(k-1) &\geqslant k(n-r)-(n-r-1)(k-1) \\ &\geqslant (n-r)+(k-1) > 0. \end{aligned}$$

This implies that A has more than $r(n-r)$ 0's, contradicting $\tau = r(n-r)$. Hence the trace of A equals n.

Let the characteristic polynomial of A be

$$\chi_A(\lambda) = \lambda^n - c_1\lambda^{n-1} + c_2\lambda^{n-2} - c_3\lambda^{n-3} + \cdots + (-1)^n c_n.$$

As already observed, $c_k \geqslant 0$ for $k=1,\dots,n$. Since the trace of A is n, we have $c_1 = n$ and, as in the proof of Lemma 2.3, $c_2 = \tau = r(n-r)$. By Lemmas 2.4 and 2.5, $c_3 > 0$ and $c_{k+1} \leqslant r_k c_k$ for $k=1,\dots,n-1$, where r_k is the number of 0's in row k of A. Since in this case $r_k < n-r$ for $k=1,\dots,n$, we conclude that

$$c_{k+1} \leqslant (n-r)c_k \qquad \text{for} \quad k=1,\dots,n-1 \tag{2.4}$$

with strict inequality when $c_k > 0$. The characteristic polynomial of A can be rewritten as

$$\chi_A(\lambda) = \lambda^{n-2}(\lambda - r)[\lambda - (n - r)] + p(\lambda),$$

where

$$\begin{aligned} p(\lambda) &= -c_3\lambda^{n-3} + c_4\lambda^{n-4} - c_5\lambda^{n-5} + c_6\lambda^{n-6} - \cdots \\ &= -\lambda^{n-4}(c_3\lambda - c_4) - \lambda^{n-6}(c_5\lambda - c_6) - \cdots . \end{aligned}$$

Since $\chi_A(n-r) = p(n-r)$, it now follows from (2.4) and $c_3 > 0$ that $\chi_A(n-r) < 0$. Thus $\rho(A) > n - r$.

Since all possibilities have been accounted for, the lemma follows. ∎

We now obtain the analogue of Lemma 2.6 when τ is not of the form $r(n-r)$. Recall that

$$\lambda(n,\tau) = \frac{n + \sqrt{n^2 - 4\tau}}{2}.$$

LEMMA 2.7. *Let the integer τ satisfy*

$$(r-1)[n-(r-1)] < \tau < r(n-r) \tag{2.5}$$

for some integer r where $2 \leqslant r \leqslant n-1$ and $n - r \geqslant r$. Let $A \in \tilde{L}(n,\tau)$, and suppose that A does not have the form (2.1) *for any nonnegative integers k and l with $k + l = n$. Then $\rho(A) > \lambda(n,\tau)$.*

Proof. The smallest n for which there exists a matrix satisfying the hypotheses of the lemma is $n = 5$, for which the conclusion is readily verified. Thus we may assume that $n > 5$, and we proceed by induction on n. The structure of the proof is similar to that of Lemma 2.6.

Case 1. *A has exactly $n-(r-1)$ 0's in row* 1. The principal submatrix $A(1|1)$ has exactly $\tau_1 = \tau - n + r - 1$ 0's. If $A(1|1)$ has the form (2.1) with $k + l = n - 1$, then by Lemma 2.3, $\rho(A(1|1)) = \lambda(n-1, \tau_1)$. If $A(1|1)$ does not have the form (2.1) but $\tau_1 = s(n-1-s)$ for some s with $n - 1 - s \geqslant s$, then by Lemma 2.6, $\rho(A(1|1)) > n - 1 - s = \lambda(n-1, \tau_1)$. Otherwise we may apply the inductive assumption and conclude that $\rho(A(1|1)) > \lambda(n-1, \tau_1)$.

Thus the conclusion will hold in this case once it is verified that

$$\lambda(n-1,\tau_1) > \lambda(n,\tau). \tag{2.6}$$

But (2.6) holds if and only if

$$\tfrac{1}{2}\left(n-1+\sqrt{(n-1)^2-4(\tau-n+r-1)}\right) > \tfrac{1}{2}\left(n+\sqrt{n^2-4\tau}\right),$$

which is easily shown equivalent to

$$\tau > (r-1)[n-(r-1)].$$

Thus (2.6) follows from (2.5), and the proof of case 1 is complete. The next case is handled in a similar way.

Case 2. *A has exactly* $n-(r-1)$ *0's in column* n.

Case 3. *A has more than* $n-r+1$ *0's in row* 1. As in case 3 of Lemma 2.6, there is a matrix $A' \in \tilde{L}(n-1,\tau_1)$ where $A' \leqslant A(1|1)$ and $\tau_1=\tau-n+r-1$, and the conclusion follows as in case 1 above. Case 4 is dealt with in a similar way.

Case 4. *A has more than* $n-r+1$ *0's in column* n.

Case 5. *A has less than* $n-r+1$ *0's in each of row* 1 *and column* n. If the trace of A did not equal n, then as in case 5 of Lemma 2.6, one shows that A would have more than $r(n-r)$ 0's, contradicting (2.5). Hence A has trace equal to n. Therefore the characteristic polynomial of A is

$$\chi_A(\lambda) = \lambda^n - c_1\lambda^{n-1} + c_2\lambda^{n-2} - c_3\lambda^{n-3} + \cdots + (-1)^n c_n,$$

where $c_1 = n$, $c_2 = \tau$, $c_3 > 0$ by Lemma 2.4, and $c_k \geqslant 0$ for $k > 3$. In addition, by Lemma 2.5, $c_{k+1} \leqslant r_k c_k$ for $k = 1,\ldots,n-1$ where r_k is the number of 0's in row k of A. Since by hypothesis $\tau < r(n-r)$ where $n-r \geqslant r$, we conclude that $\lambda(n,\tau) > n-r$, so that in this case $r_k \leqslant n-r < \lambda(n,\tau)$ for $k=1,\ldots,n$. Hence

$$c_{k+1} \leqslant \lambda(n,\tau)c_k \qquad \text{for} \quad k=1,\ldots,n-1, \tag{2.7}$$

with strict inequality when $c_k > 0$. We write

$$\chi_A(\lambda) = \lambda^{n-2}(\lambda^2 - n\lambda + \tau) + p(\lambda),$$

where

$$p(\lambda) = -\lambda^{n-4}(c_3\lambda - c_4) - \lambda^{n-6}(c_5\lambda - c_6) - \cdots .$$

Since $\chi_A(\lambda(n,\tau)) = p(\lambda(n,\tau))$, it now follows from (2.7) and $c_3 > 0$, that $\chi_A(\lambda(n,\tau)) < 0$. Hence $\rho(A) > \lambda(n,\tau)$. ■

In summary thus far, we obtain the following

LEMMA 2.8. *Let τ be an integer with $0 \leqslant \tau \leqslant \lfloor n/2 \rfloor \lceil n/2 \rceil$, and let $A \in \tilde{L}(n,\tau)$. Then $\rho(A) \geqslant \lambda(n,\tau)$ with equality if and only if there are nonnegative integers k and l with $k + l = n$ such that A has the form* (2.1).

Proof. This lemma is an immediate consequence of Lemmas 2.3, 2.6, and 2.7. ■

We will now complete the proof of Theorem 2.1 by applying the results (1.6) and (1.7) of Schwarz.

Proof of Theorem 2.1. By (1.7), Lemma 2.3, and Lemma 2.8, we need only show that if A is a matrix in $L(n,\tau) \setminus \tilde{L}(n,\tau)$ having the property that there does not exist a permutation matrix P such that P^tAP has the form (2.1) with $k + l = n$, then $\rho(A) > \lambda(n,\tau)$. Consider such a matrix A, and let $A_0 = Q^tAQ, A_1, \dots, A_s = B$ be matrices in $L(n,\tau)$ satisfying (1.6). Hence $\rho(A) = \rho(Q^tAQ) \geqslant \rho(A_i) \geqslant \rho(B)$ for $i = 1,\dots, s-1$, and since $B \in \tilde{L}(n,\tau)$, $\rho(B) \geqslant \lambda(n,\tau)$ by Lemma 2.8. If B does not have the form (2.1), then by Lemma 2.8 again, $\rho(B) > \lambda(n,\tau)$ and hence $\rho(A) > \lambda(n,\tau)$. Thus we may assume B has the form (2.1). It now follows from (ii) of (1.6) that there exists an integer j such that $F = A_{j+1}$ has the form (2.1) with $k = r$ but $E = A_j$ does not have the form (2.1) for any k; moreover, F is obtained from E by switching a 0 and a 1 in E where the 0 immediately precedes the 1 in some row or immediately follows the 1 in some column. The two possibilities are similar, and we only consider the first. Hence

$$E = \begin{bmatrix} D & X' \\ J_{n-r,r} & J_{n-r} \end{bmatrix}, \qquad F = \begin{bmatrix} J_r & X \\ J_{n-r,r} & J_{n-r} \end{bmatrix},$$

where for some i between 1 and r, X' is obtained from X by replacing a 0 in its $(i,1)$ position with a 1 and D is obtained from J_r by replacing the 1 in its (i,r) position with a 0.

Case 1. $X \neq 0$. It then follows easily that both E and F are irreducible. By (1.1) there is a positive vector x such that

$$Ex = \rho(E)x. \tag{2.8}$$

Since E does not have the form (2.1) for any k, in particular for $k = r - 1$, it follows that the last row of X' contains a 0. Since row $r + 1$ of E contains no 0's, the rth and $(r+1)$st equations of (2.8) imply that $x_r < x_{r-1}$. It now follows from (2.8) that

$$Fx \leqslant \rho(E)x, \qquad F(x) \neq \rho(E)x.$$

Since F is irreducible, we now conclude using (1.3) that $\rho(F) < \rho(E)$. Hence

$$\rho(A) \geqslant \rho(E) > \rho(F) \geqslant \rho(B) \geqslant \lambda(n, \tau),$$

and the desired conclusion holds in this case.

Case 2. $X = 0$. In this case $\tau = r(n - r)$ and $\lambda(n, \tau) = \max\{r, n - r\}$. Moreover $F = B$. First suppose that $n - r \geqslant r$ so that $\lambda(n, \tau) = n - r$. The matrix E has an irreducible principal submatrix M of order $n - r + 1$ equal to

$$\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & & & & \\ \vdots & & & J_{n-r} & \\ 1 & & & & \end{bmatrix},$$

which by (1.2) satisfies $\rho(M) > n - r$. Now using (1.5) we conclude that

$$\rho(A) \geqslant \rho(E) \geqslant \rho(M) > n - r.$$

Now suppose $r > n - r$ so that $\lambda(n, \tau) = r$. Now E has an irreducible principal submatrix M of order $r + 1$ equal to

$$\left[\begin{array}{c|cc} & 1 & 0 \\ & \vdots & \vdots \\ & 1 & 0 \\ J_{r+1,r-1} & 0 & 1 \\ & 1 & 0 \\ & \vdots & \vdots \\ & 1 & 0 \\ & 1 & 1 \end{array}\right],$$

which by (1.2) satisfies $\rho(M) > r$. Using (1.5), we now conclude that $\rho(A) > r$. Hence $\rho(A) > \lambda(n, \tau)$ holds in this case also, completing the proof of Theorem 2.1. ∎

3. BOUNDS FOR THE MINIMUM SPECTRAL RADIUS

Let n be a positive integer and let τ be an integer with $0 \leqslant \tau \leqslant n^2$. In this section we determine the minimum spectral radius $\tilde{\rho}(n, \tau)$ for a certain sequence of values of τ. We then use this result in order to bound $\tilde{\rho}(n, \tau)$ in general between two consecutive integers.

Let k be an integer with $1 \leqslant k \leqslant n$, and write $n = qk + l$ where q is a positive integer and $0 \leqslant l < k$. We define $A_{n,k}$ to be the $n \times n$ block triangular matrix

$$A_{n,k} = \begin{bmatrix} J_k & & & 0 \\ & \ddots & & \\ & & J_k & \\ & J & & J_l \end{bmatrix}$$

which has q diagonal blocks equal to J_k, one diagonal block equal to J_l, and all 1's below the diagonal blocks. The number of 0's of A is given by

$$\tau_{n,k} = \frac{q(q-1)}{2} k^2 + qkl.$$

Note that $A_{n,k} \in \tilde{L}(n, \tau_{n,k})$.

We define two sequences $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ as follows. If i is at most equal to the number of diagonal blocks of $A_{n,k}$, we define x_i to be the number of positions in the ith diagonal block of $A_{n,k}$ which have a 0 in $A_{n,k-1}$; otherwise we define $x_i = 0$. Similarly, if i is at most equal to the number of diagonal blocks of $A_{n,k-1}$, we define y_i to be the number of positions in the ith diagonal block of $A_{n,k-1}$ which have a 0 in $A_{n,k}$. One checks that

$$y_1 = 0,$$

$$x_1 - y_2 \geqslant 1,$$

$$x_i - y_{i+1} \geqslant 0 \qquad \text{for} \quad i = 2, 3, \ldots.$$

Moreover, for $2 \leqslant k \leqslant n$,

$$\begin{aligned}\tau_{n,k-1} - \tau_{n,k} &= \sum_{i \geqslant 1} x_i - \sum_{i \geqslant 1} y_i \\ &= \sum_{i \geqslant 1} (x_i - y_{i+1}) \geqslant 1.\end{aligned}$$

Hence

$$0 = \tau_{n,n} < \tau_{n,n-1} < \cdots < \tau_{n,1} = \binom{n}{2}. \tag{3.1}$$

We first obtain the minimum spectral radius when the number of 0's is $\tau_{n,k}$. A block triangular matrix with q blocks is called a *q-block triangular matrix*.

THEOREM 3.1. *Let k be an integer with $1 \leqslant k \leqslant n$, and write $n = qk + l$ where $q \geqslant 1$ and $0 \leqslant l < k$. Let τ be a nonnegative integer with $\tau \leqslant \tau_{n,k}$. Then*

$$\tilde{\rho}(n, \tau) \geqslant k.$$

Moreover, for $A \in L(n, \tau)$, $\rho(A) = k$ if and only if $\tau = \tau_{n,k}$ and there is a permutation matrix P such that P^tAP is a q-block triangular matrix where

(3.2) *One block is a $(k + l) \times (k + l)$ matrix with kl 0's of the form*

$$\begin{bmatrix} J_r & X \\ J_{s,r} & J_s \end{bmatrix}$$

for some r and s with $r + s = k + l$,

(3.3) *the remaining diagonal blocks are equal to J_k, and*

(3.4) *all entries below the diagonal blocks equal* 1.

Proof. Using Lemma 2.3, we see that a matrix of the form (3.2) has spectral radius equal to k, and it follows that any q-block triangular matrix satisfying (3.2), (3.3), and (3.4) belongs to $L(n, \tau_{n,k})$ and has spectral radius equal to k. In particular, by combining the last two blocks of $A_{n,k}$, $A_{n,k}$ is a

q-block triangular matrix satisfying (3.2), (3.3), and (3.4). Hence $\tilde{\rho}(n, \tau_{n,k}) \leqslant k$. Let $\tau \leqslant \tau_{n,k}$. It remains to show that if $A \in L(n, \tau)$, then $\rho(A) \geqslant k$ with equality only when $\tau = \tau_{n,k}$ and A satisfies the conditions given in the theorem. We first prove by induction on q with l fixed, that when $A \in \tilde{L}(n, \tau)$, $\rho(A) \geqslant k$ with equality only if $\tau = \tau_{n,k}$ and A is a q-block triangular matrix satisfying (3.2), (3.3), and (3.4).

Let $A \in \tilde{L}(n, \tau)$. If $q = 1$, then $n = k + l$ and $\tau \leqslant kl \leqslant \lfloor n/2 \rfloor \lceil n/2 \rceil$, and the conclusion holds by Theorem 2.1. Let $q \geqslant 2$. for $j = 1, \dots, n$, define A_j to be the principal submatrix $A[1, \dots, j|1, \dots, j]$ of A. If for some j we have $\rho(A_j) > k$, then by (1.5) $\rho(A) > k$. Hence we may suppose that $\rho(A_j) \leqslant k$ for $j = 1, \dots, n$. It now follows from Theorem 2.1 that

(3.5) A_j has at least $k(j-k)$ 0's and exactly $k(j-k)$ 0's only when $\rho(A_j) = k$ $(j = k, k+1, \dots, n)$.

First assume that all the 0's of A_{2k} lie in the first k rows. Then $A[k+1, \dots, 2k|k+1, \dots, 2k] = J_k$. Suppose that $A[1, \dots, k|k+1, \dots, 2k] \neq 0$. Then it follows that A_{2k}, and hence A, has an irreducible principal submatrix B of order $k+1$, which in turn has a principal submatrix equal to J_k. Therefore by two applications of (1.5), $\rho(A) \geqslant \rho(B) > \rho(J_k) = k$. Hence we may suppose $A[1, \dots, k|k+1, \dots, 2k] = 0$. Since $A \in \tilde{L}(n, \tau)$ we therefore have $A[1, \dots, k|k+1, \dots, n] = 0$. It now follows that the matrix $C = A[k+1, \dots, n|k+1, \dots, n]$ has τ' 0's where

$$\tau' \leqslant \tau_{n,k} - k(n-k) = \tau_{n-k,k}.$$

Moreover, C has exactly $\tau_{n-k,k}$ 0's if and only if $\tau = \tau_{n,k}$ and $A[1, \dots, k|1, \dots, k] = J_k$. Since $C \in \tilde{L}(n-k, \tau')$, we may apply the inductive assumption to C to conclude that $\rho(C) \geqslant k$ with equality only if $\tau' = \tau_{n-k,k}$ and C is a $(q-1)$-block triangular matrix satisfying (3.2), (3.3), and (3.4). If $\tau' < \tau_{n-k,k}$ then $\rho(C) > k$ and therefore $\rho(A) > k$. Hence we may suppose that $\tau' = \tau_{n-k,k}$. It now follows that $\tau = \tau_{n,k}$ and

$$A = \begin{bmatrix} J_k & 0 \\ J_{n-k,k} & C \end{bmatrix}; \tag{3.6}$$

hence A is a q-block triangular matrix satisfying (3.2), (3.3), and (3.4).

Now assume that A_{2k} has a 0 in one of rows $k+1, \dots, 2k$. Let $i+1$ be the smallest integer such that A_{i+1} has a 0 below row k. Then $i+1 \leqslant 2k$, A_i has all its 0's in rows $1, \dots, k$, and A_{i+1} contains a 0 below row k in column

$i+1$. Since $A \in \tilde{L}(n,\tau)$, it now follows that $A[1,\dots,k|i+1,\dots,n]=0$. Using (3.5), we now conclude that A has at least $k(i-k)+k(n-i)=k(n-k)$ 0's in rows $1,\dots,k$, and exactly $k(n-k)$ 0's only when $\rho(A_i)=k$. Again let $C=A[k+1,\dots,n|k+1,\dots,n]$, and let τ' be the number of 0's of C. Then $\tau' \leqslant \tau_{n-k,k}$ with equality if and only if $\tau=\tau_{n,k}$ and A has exactly $k(n-k)$ 0's in the first k rows. Applying the inductive assumption to C and using (1.5), we conclude $\rho(A)\geqslant\rho(C)\geqslant k$, and $\rho(A)=k$ only when $\tau'=\tau_{n-k,k}$ and C is a $(q-1)$-block triangular matrix satisfying (3.2), (3.3), and (3.4). Suppose that $\rho(A)=k$. Then $\tau'=\tau_{n-k,k}$, from which we now conclude that $\tau=\tau_{n,k}$, A has exactly $k(n-k)$ 0's in its first k rows, and $\rho(A_i)=k$. Since $A[1,\dots,k|i+1,\dots,n]=0$ and A_i has no 0's below row k, A_i has exactly $k(n-k)-k(n-i)=k(i-k)$ 0's.

Case 1. A_i *is reducible.*. By Theorem 2.1, the 0's of A_i are contained in an $r\times(i-r)$ submatrix D. Since A_i is reducible, $D=0$. Since A_i has exactly $k(i-k)$ 0's, either $r=k$ or $r=i-k$. If $r=k$, then

$$A_i=\begin{bmatrix} J_k & 0 \\ J_{i-k,k} & J_{i-k}\end{bmatrix},$$

and hence the principal submatrix A_k of A equals J_k. Thus A has the form (3.6), and we conclude that A is a q-block triangular matrix satisfying (3.2), (3.3), and (3.4). Let $r=i-k$. Then

$$A_i=\begin{bmatrix} J_{i-k} & 0 \\ J_{k,i-k} & J_k\end{bmatrix}.$$

Suppose the $(i,i+1)$-entry of A was 1. Then by (1.5),

$$\rho(A)\geqslant\rho(A[i-k+1,\dots,i+1|i-k+1,\dots,i+1])>\rho(J_k)=k.$$

Hence we may suppose that the $(i,i+1)$-entry of A is 0, and since $A\in\tilde{L}(n,\tau)$, that $A[1,\dots,i|i+1,\dots,n]=0$. Hence C has the form

$$\begin{bmatrix} J_{i-k} & 0 \\ J_{n-i,i-k} & Y\end{bmatrix}.$$

Since $i-k<k$ and since C is a $(q-1)$-block triangular matrix satisfying (3.2), (3.3), (3.4), it now follows that the block of C satisfying (3.2) is

$C[1,\ldots,k+l|1,\ldots,k+l]$. Hence

$$C=\left[\begin{array}{cc|ccc} J_{i-k} & 0 & & 0 & \\ J_{t,i-k} & J_t & & & \\ \hline & & J_k & & 0 \\ \multicolumn{2}{c|}{J_{n-k-l,k+l}} & & \ddots & \\ & & J & & J_k \end{array}\right], \qquad t=2k+l-i. \tag{3.7}$$

Since the matrix $C[1,\ldots,k+l|1,\ldots,k+l]$ satisfying (3.2) has spectral radius equal to k by Lemma 2.3, it follows that $k=t=2k+l-i$ and hence $i=k+l$. Thus A has the form

$$\left[\begin{array}{cc|ccc} J_l & 0 & & 0 & \\ J_{k,l} & J_k & & & \\ \hline & & J_k & & 0 \\ \multicolumn{2}{c|}{J_{n-k-l,k+l}} & & \ddots & \\ & & J & & J_k \end{array}\right],$$

and A is a q-block triangular matrix satisfying (3.2), (3.3), and (3.4).

Case 2. *A_i is irreducible.* Suppose the $(i,i+1)$-entry of A was 1. Then by (1.5)

$$\rho(A)\geqslant\rho(A_{i+1})>\rho(A_i)=k.$$

Hence we may suppose that the $(i,i+1)$-entry of A is 0, and so $A[1,\ldots,i|i+1,\ldots,n]=0$. As in case 1, we obtain that $i=k+l$ and C has the form (3.7) with $t=k$. Since $\rho(A_i)=k$ and A_i has exactly $k(i-k)=kl$ 0's, it follows from Corollary 2.2 that A_i is of the form (3.2) for some r. Hence A is a q-block triangular matrix satisfying (3.2), (3.3), and (3.4). This completes the proof when $A\in\tilde{L}(n,\tau)$.

Now let $A\in L(n,\tau)$. Then it follows from (1.7) and the validity of the theorem for $\tilde{L}(n,\tau)$ that $\rho(A)\geqslant k$ with equality only when $\tau=\tau_{n,k}$. Suppose $A\in L(n,\tau_{n,k})\backslash\tilde{L}(n,\tau_{n,k})$ and there does not exist a permutation matrix P such that P^tAP is a q-block triangular matrix satisfying (3.2), (3.3), and (3.4). We show that $\rho(A)>k$, thereby completing the proof of the theorem. Let Q be a permutation matrix such that $A_0=Q^TAQ$, $A_1,\ldots,A_s=B$ satisfies (1.6). If B is not a q-block triangular matrix satisfying (3.2), (3.3), and (3.4), then it follows from what we have proved above and (iii) of (1.6) that $\rho(A) \geqslant \rho(B) > k$. Hence we may assume that B is a q-block triangular matrix satisfying (3.2), (3.3), and (3.4). Using (ii) of (1.6), we now conclude that there is an integer j such that $F = A_{j+1}$ is a q-block triangular matrix satisfying (3.2), (3.3), and (3.4) but $E = A_j$ is not. Moreover F is obtained from E by switching an entry $\alpha = 0$ with an entry $\beta = 1$ in E where α immediately precedes β in some row, or immediately follows β in some column. The two cases are similar, and we only consider the first.

First suppose that α and β lie in a row which meets a J_k block of F. Then E has an irreducible principal submatrix M of order $k+1$ equal to

$$\left[\begin{array}{ccc} & 1 & 0 \\ & \vdots & \vdots \\ & 1 & 0 \\ & 0 & 1 \\ J_{k+1,k-1} & 1 & 0 \\ & \vdots & \vdots \\ & 1 & 0 \\ & 1 & 1 \end{array}\right],$$

which by (1.2) satisfies $\rho(M) > k$. It now follows from (1.5) that $\rho(A) > k$.

Now suppose that α and β lie in a row which meets the block N of F of order $k + l$ specified in (3.2). Therefore

(3.8) $$N = \begin{bmatrix} J_r & X \\ J_{s,r} & J_s \end{bmatrix},$$

where $r + s = k + l$. Let N' be the submatrix of E corresponding to N. First assume that α and β are both contained in N'. Hence both N and N' have kl 0's. Then E is a q-block triangular matrix, and since E is not a q-block triangular matrix satisfying (3.2), (3.3), and (3.4), N' does not have the form

(3.9) $$\begin{bmatrix} J_u & Y \\ J_{v,u} & J_v \end{bmatrix}$$

for any choice of u and v with $u + v = k + l$. But then the form (3.8) of N implies that α is the (i, r)-entry of N' for some i with $i \leqslant r$. Since N' does not have the form (3.9) with $u = r - 1$, there exists a j with $j > r$ such that the (r, j)-entry γ of N' is 0. Suppose there is a permutation matrix R such that $R^t N' R$ has the form (3.9) where $u + v = k + l$. Let σ be the permutation of $\{1, \dots, k + l\}$ corresponding to R. Considering α, we conclude that $\sigma(r) > u$, while considering γ, we conclude that $\sigma(r) \leqslant u$. Hence no such R exists. It follows from Corollary 2.2 and (1.5) that

$$\rho(A) \geqslant \rho(E) \geqslant \rho(N') > k.$$

The remaining possibility is that α is contained in N' but β is not. Then E has a principal submatrix of the form

$$\begin{bmatrix} N' & Z \\ J_{k,k+l} & J_k \end{bmatrix}, \tag{3.10}$$

where Z contains β but is otherwise equal to 0. The matrix in (3.10) has an irreducible principal submatrix M of order $k + 1$ with column sums $a, k + 1, k, \dots, k$, where $a \geqslant k$. Using (1.2) and (1.5) we obtain

$$\rho(A) \geqslant \rho(E) \geqslant \rho(M) > k.$$

Therefore $\rho(A) > k$ whenever $A \in L(n, \tau_{n,k}) \setminus \tilde{L}(n, \tau_{n,k})$ does not satisfy the conditions in the theorem, and the proof of the theorem is complete. ■

Using Theorem 3.1 and the theory of nonnegative matrices, we now obtain the second main result of this section.

THEOREM 3.2. *Let n be a positive integer, and let τ be an integer with*

$$0 \leqslant \tau < \binom{n}{2}.$$

Choose an integer k between 1 and $n - 1$ such that

$$\tau_{n,k+1} \leqslant \tau < \tau_{n,k}. \tag{3.11}$$

Then

$$k < \tilde{\rho}(n, \tau) \leqslant k + 1.$$

Proof. It follows from (3.1) that there is a unique integer k such that (3.11) holds. Using Theorem 3.1 and (1.4) we see that

$$k < \tilde{\rho}(n, \tau) \leqslant \tilde{\rho}(n, \tau_{n,k+1}) = k + 1. \qquad ■$$

We note that it is possible that $\tilde{\rho}(n,\tau)=k+1$ for $\tau>\tau_{n,k+1}$. For example, using (1.7) it is easy to see that $\tilde{\rho}(4,5)=2$, yet $5>\tau_{4,2}=4$.

To conclude we now consider those values of τ not included in Theorem 3.2. We first note that if $A\neq 0$ is an irreducible matrix of 0's and 1's, then by (1.2) $\rho(A)\geqslant 1$ with equality if and only if A is a permutation matrix. Let τ be an integer with

$$\binom{n}{2}\leqslant\tau<\binom{n+1}{2},$$

and let $A\in L(n,\tau)$. Then at least one of the irreducible components of A in its Frobenius normal form [1, 5] is a nonzero matrix, and hence $\rho(A)\geqslant 1$. Moreover, $\rho(A)=1$ if and only if each irreducible component of A is a permutation matrix. In particular, we have that

$$\tilde{\rho}(n,\tau)=1\qquad\text{for}\quad\binom{n}{2}\leqslant\tau<\binom{n+1}{2}.$$

Now let τ be an integer with

$$\binom{n+1}{2}\leqslant\tau\leqslant n^2.$$

Then it follows that $\tilde{\rho}(n,\tau)=0$ and that for $A\in L(n,\tau)$, $\rho(A)=0$ if and only if each irreducible component of A is a 1×1 zero matrix.

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Received 22 June 1985; revised 2 December 1985